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LETTER TO THE EDITOR

Self-adjoint extensions of the Laplacian and Aharonov–Bohm operators with a potential supported on a circle

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Abstract
Using the definition of a self-adjoint operator, we give the self-adjoint extensions of the Laplacian and Aharonov–Bohm operators with a potential supported on a circle of radius R. This study gives rise to interesting results, namely all self-adjoint extensions are given by a four-parameter family of self-adjoint operators, reproducing a matrix representation for $U(2)$ symmetry.

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An extremely useful mathematical framework for dealing with quantum theory is the operator analysis in Hilbert space. In this context, the quantization of operators plays a major role. In the present letter, attention will be paid to the quantization of Aharonov–Bohm operator with an additional potential supported on a circle of radius R. The Aharonov–Bohm effect has received much attention in recent years [1–12]. This phenomenon is based on the fact that there are measurable effects that can be attributed directly to the electromagnetic vector potential and only non-locally to the magnetic field itself [1]. Numerous potential applications, for instance in superconductivity, and the fundamental significance of electromagnetic potentials in the quantum theory motivate the interest in this phenomenon.

As far as the theoretical work is concerned, it is worth knowing that in [11], Exner *et al* provide a remarkable study on the most generalized boundary conditions for the Aharonov–Bohm flux intersecting the plane at the origin on the background of a homogeneous magnetic field. They use the standard techniques based on self-adjoint extensions and find a four-parameter family of boundary conditions; the other two parameters of the model are the Aharonov–Bohm flux and the homogeneous magnetic field. These generalized boundary conditions may be regarded as a combination of the Aharonov–Bohm effect with a point interaction. In an earlier work [5], Dabrowski and Šťovíček investigate a five-parameter family of Hamiltonian operators which describe a quantum particle interacting with a thin solenoid and a magnetic flux. One of the five parameters is just the value of the flux and the other four correspond to the strength of a singular potential barrier (sort of a combination of Dirac δ and δ') and can be interpreted as penetrability coefficients of the shielded solenoid. A general operator of this family corresponds to an intricate mixture between the Aharonov–Bohm effect and the point interactions which is manifested more concretely via the mixing between the angular and the radial boundary conditions. For details, see [5].

More recently, Exner and Tater [12] discuss of a ring-shaped soft quantum wire modelled by δ interaction supported by the ring of a generally nonconstant coupling strength. They investigate spectral properties of a two-dimensional quantum particle subject to a δ interaction supported by a circle ring, plus possibly a magnetic field. In particular, in this work, these authors illustrate that it is rather the geometry of the interaction curve than its topology which determines the spectral properties. They also consider the situation when the coupling strength is randomly varying and when the particle is exposed to a magnetic field perpendicular to the ring plane.

In this letter, we provide a study based on a straightforward and a 'very natural' theoretical construction of the self-adjoint extensions of the Aharonov–Bohm operator H_α with an interaction potential V supported on a circle of radius R, defined as follows:

$$H_B = H_\alpha + V. \tag{1}$$

Taking the origin of the coordinate system at the position of the solenoid and introducing the vector potential $A(x, y) = -\alpha c e^{-1}(-yr^{-2}, xr^{-2})$, where $r^2 = (x^2 + y^2)$ and $-2\pi c\alpha e^{-1}$ is the magnetic flux through the solenoid, the formal free Aharonov–Bohm operator H_α written in polar coordinates reads

$$H_\alpha = -\frac{\partial^2}{\partial r^2} - r^{-1}\frac{\partial}{\partial r} + r^{-2}\left(\mathrm{i}\frac{\partial}{\partial\phi} - \alpha\right)^2. \tag{2}$$

In (2), we have fixed $\hbar = 1, m = 1/2$. Moreover, without loss of generality, we suppose $0 < \alpha < 1$.

Using the partial wave expansion, the Hamiltonian (2) can be reduced to each subspace of fixed angular momentum and a complete set of eigenfunctions can be constructed. All the possible self-adjoint (s.a.) extensions of this operator defined on $C_0^\infty(\mathbb{R}^2\backslash\{O\})$ are also investigated by Adami and Teta [6]. Since the deficiency indices are (2, 2), there is a family of s.a. extensions parametrized by the unitary map U from one deficiency subspace to the other. Since the subspaces have two dimensions, the parametrization involves four real parameters. See [6] and references therein for more details on such parametrizations.

The radial part corresponding to the free Aharonov–Bohm operator H_α in (1) reads [1]

$$h_{\alpha m} := \left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha+m)^2 - \frac{1}{4}}{r^2}\right]\Bigg|_{\{C_0^\infty(\mathbb{R}^2\backslash\{\partial\overline{\Gamma(O,R)}\})\}} \tag{3}$$

where $\overline{\Gamma(O, R)}$ is a closed circle of radius R centred at the origin of $\mathbb{R}^2$. Each of the extensions of the operator will be characterized by a specific behaviour, i.e. boundary conditions for the elements of the domain near $r = R$.

Let us also consider the non-negative Laplacian operator (obtained by setting $(\alpha+m)^2 - \frac{1}{4} = 0$ in (3))

$$H_0 = -\Delta|_{\{C_0^\infty(\mathbb{R}^2\backslash\{\partial\overline{\Gamma(O,R)}\})\}} := -\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Bigg|_{\{C_0^\infty(\mathbb{R}^2\backslash\{\partial\overline{\Gamma(O,R)}\})\}}. \tag{4}$$

Then let us consider the closed and non-negative operators $\dot{H}_0$ and $\dot{H}_\alpha$, respectively:

$$\dot{H}_0 = \overline{H_0|_{\{C_0^\infty(\mathbb{R}^2\setminus\{\partial\overline{\Gamma(O,R)}\})\}}} \tag{5}$$

with the domain

$$D(\dot{H}_0) = \left\{u \in L^2(\mathbb{R}^2) \cap H^{2,2}_{loc}(\mathbb{R}^2)/u(\partial\overline{\Gamma(O,R)}) = 0, H_0 u \in L^2(\mathbb{R}^2)\right\} \tag{6}$$

$$\dot{H}_\alpha = \overline{H_\alpha|_{\{C_0^\infty(\mathbb{R}^2\setminus\{\partial\overline{\Gamma(O,R)}\})\}}} \tag{7}$$

with the domain

$$D(\dot{H}_\alpha) = \left\{u \in L^2(\mathbb{R}^2) \cap H^{2,2}_{loc}(\mathbb{R}^2)/u(\partial\overline{\Gamma(O,R)}) = 0, H_\alpha u \in L^2(\mathbb{R}^2)\right\} \tag{8}$$

where $H^{m,n}_{loc}(\Omega)$ is the local Sobolev space of indices (m, n).

Decomposing the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$ with respect to angular momenta, i.e. introducing spherical coordinates (with centre R), we obtain

$$L^2(\mathbb{R}^2) = L^2(\mathbb{R}^+) \bigotimes L^2(S^1) \tag{9}$$

where S^1 is the unit circle in $\mathbb{R}^2$. Using now, in addition, the isomorphism U in order to remove the weight factor r from the measure:

$$U : \begin{cases} L^2((0,\infty); r\,\mathrm{d}r) \longrightarrow L^2((0,\infty); \mathrm{d}r) \equiv L^2((0,\infty)) \\ u \longmapsto (Uu)(r) = \sqrt{r}u(r) \end{cases} \tag{10}$$

we can express (9) as

$$L^2(\mathbb{R}^2) = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1}(L^2(\mathbb{R}^+)) \bigotimes \left[\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}\right] \qquad m \in \mathbb{Z}. \tag{11}$$

With respect to this decomposition, $\dot{H}_\alpha$ equals the direct sum

$$\dot{H}_\alpha = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1}\dot{h}_{\alpha,m}U \bigotimes \mathbb{1} \tag{12}$$

where the operator $\dot{h}_{\alpha,m}$ in $L^2(]0,\infty[)$ is defined by

$$\dot{h}_{\alpha,m} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha+m)^2 - \frac{1}{4}}{r^2} \qquad r > 0 \tag{13}$$

with the domain

$$\begin{aligned}\mathcal{D}(\dot{h}_{\alpha,m}) = \big\{u \in L^2(]0,\infty[, \mathrm{d}r) \cap H^{2,2}_{loc}(]0,\infty[); u(0_+) = 0 \text{ if } (\alpha+m)^2 - \tfrac{1}{4} = 0;\\ u_\pm = 0; -u'' + \left((\alpha+m)^2 - \tfrac{1}{4}\right)r^{-2}u \in L^2((0,\infty))\big\} \qquad m \in \mathbb{Z}.\end{aligned} \tag{14}$$

Here and in the following, we set $u_\pm := \lim_{\epsilon\longrightarrow 0} u(R\pm\epsilon)$.

Note that, when $(\alpha+m)^2 - \frac{1}{4} = 0$, we recover the operator

$$\dot{h}_0 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} \tag{15}$$

with the domain

$$\mathcal{D}(\dot{h}_0) = \left\{u \in L^2(]0,\infty[, \mathrm{d}r) \cap H^{2,2}_{loc}(]0,\infty[); u(0_+) = 0; u_\pm = 0; -u'' \in L^2((0,\infty))\right\}. \tag{16}$$

The adjoint operator $\dot{h}_0^*$ of $\dot{h}_0$ is then given by

$$\dot{h}_0^* = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$

with the domain

$$D(\dot{h}_0^*) = \Big\{ u \in L^2(]0,\infty[,\mathrm{d}r) \cap H^{2,2}_{loc}(]0,\infty[-\{R\});\, u(0_+) = 0;$$
$$u_+ = u_- \equiv u(R);\, -\frac{\mathrm{d}^2}{\mathrm{d}r^2}u \in L^2(]0,\infty[) \Big\}. \tag{17}$$

The adjoint operator $\dot{h}^*_{\alpha,m}$ of $\dot{h}_{\alpha,m}$ is defined by

$$\dot{h}^*_{\alpha,m} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha+m)^2 - \frac{1}{4}}{r^2}$$

with the domain

$$D(\dot{h}^*_{\alpha,m}) = \Big\{ u \in L^2(]0,\infty[,\mathrm{d}r) \cap H^{2,2}_{loc}(]0,\infty[-\{R\});\, u(0_+) = 0 \text{ if } (\alpha+m)^2 - \frac{1}{4} = 0;$$
$$u_+ = u_- \equiv u(R);\, \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha+m)^2 - \frac{1}{4}}{r^2} \right) u \in L^2(]0,\infty[) \Big\} \qquad m \in \mathbb{Z}. \tag{18}$$

From (12), we obtain

$$\dot{H}^*_\alpha = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1}\dot{h}^*_{\alpha,m}U \bigotimes \mathbb{1}. \tag{19}$$

Let us now restrict our analysis to the subspace of $D(h_{\alpha,m})$ in which the investigated operator $h_{\alpha,m}$ is self-adjoint, i.e in which for any u, v, we are in a position to define the inner product with the property:

$$\langle v | h_{\alpha,m} u \rangle = \langle h_{\alpha,m} v | u \rangle \tag{20}$$

associated with the boundary conditions expressed through an invertible matrix $\mathcal{M}$ such that

$$\begin{pmatrix} u_+ \\ u'_+ \end{pmatrix} = \mathcal{M} \begin{pmatrix} u_- \\ u'_- \end{pmatrix} \tag{21}$$

where

$$\mathcal{M} := \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \qquad \mathcal{M}_{ij} \in \mathbb{C}.$$

Thus, relation (21) takes the form

$$u_+ = \mathcal{M}_{11}u_- + \mathcal{M}_{12}u'_- \qquad u'_+ = \mathcal{M}_{21}u_- + \mathcal{M}_{22}u'_-. \tag{22}$$

The property (20) implies

$$-v^\star_+ u'_+ + v^\star_- u'_- - u_- v^{\star\prime}_- + u_+ v^{\star\prime}_+ = 0. \tag{23}$$

Using the boundary conditions (22), equation (23) can be written as

$$\left\{ -v^\star_+\mathcal{M}_{21} + v^{\star\prime}_+\mathcal{M}_{11} - v^{\star\prime}_- \right\} u_- + \left\{ -v^\star_+\mathcal{M}_{22} + v^{\star\prime}_+\mathcal{M}_{12} + v^\star_- \right\} u'_- = 0 \tag{24}$$

that gives the following system of equations

$$-v^\star_+\mathcal{M}_{21} + v^{\star\prime}_+\mathcal{M}_{11} - v^{\star\prime}_- = 0 \qquad -v^\star_+\mathcal{M}_{22} + v^{\star\prime}_+\mathcal{M}_{12} + v^\star_- = 0. \tag{25}$$

For convenience, let us put this system in a matrix form

$$\begin{pmatrix} v_- \\ v'_- \end{pmatrix} = \mathcal{N} \begin{pmatrix} v_+ \\ v'_+ \end{pmatrix} \tag{26}$$

where the matrix $\mathcal{N}$ defined by

$$\mathcal{N} := \begin{pmatrix} \overline{\mathcal{M}}_{22} & -\overline{\mathcal{M}}_{12} \\ -\overline{\mathcal{M}}_{21} & \overline{\mathcal{M}}_{11} \end{pmatrix} \tag{27}$$

has to give the matrix $\mathcal{M}^{-1}$. Hence, the matrix elements $\mathcal{M}_{ij}$ satisfy the property

$$\overline{\mathcal{M}}_{ij} = \frac{1}{\det \mathcal{M}} \mathcal{M}_{ij} \tag{28}$$

giving the conditions

$$\mathcal{M}_{ij} = |\mathcal{M}_{ij}| \exp(\mathrm{i}\phi/2) \qquad \det \mathcal{M} = \exp(\mathrm{i}\phi) \qquad |\det \mathcal{M}||\det \mathcal{N}| = 1. \tag{29}$$

Therefore, the matrices $\mathcal{M}$ realize the group $U(2)$. Then, we obtain

Theorem 1. *Let the matrices $\mathcal{M}_{2\times 2}$ and $\mathcal{N}_{2\times 2}$ be involved in relations (21) and (27) with the properties (28) and (29). Then*

(i) All self-adjoint (s.a.) extensions of the operator $\dot{h}_0$ are given by a four-parameter family of (s.a.) operators given by

$$h_{0,\mathcal{M}_{2\times 2}} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} \tag{30}$$

with the domain

$$D(h_{0,\mathcal{M}_{2\times 2}}) = \Bigg\{ u \in L^2(]0,\infty[, \mathrm{d}r) \cap H^{2,2}_{loc}(]0,\infty[-\{R\}); u(0_\pm) = 0; \begin{pmatrix} u_+ \\ u'_+ \end{pmatrix} = \mathcal{M} \begin{pmatrix} u_- \\ u'_- \end{pmatrix}; -\frac{\mathrm{d}^2}{\mathrm{d}r^2} u \in L^2(]0,\infty[) \Bigg\}. \tag{31}$$

(ii) All self-adjoint extensions of the operator $\dot{h}_{\alpha,m}$ are given by a four-parameter family of (s.a.) operators given by

$$h_{\alpha,m,\mathcal{M}_{2\times 2}} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha+m)^2 - \frac{1}{4}}{r^2} \qquad r > 0 \tag{32}$$

with the domain

$$D(h_{\alpha,m,\mathcal{M}_{2\times 2}}) = \Bigg\{ u \in L^2(]0,\infty[, \mathrm{d}r) \cap H^{2,2}_{loc}(]0,\infty[-\{R\}); u(0_+) = 0 \text{ if } (\alpha+m)^2 - \frac{1}{4} = 0; \begin{pmatrix} u_+ \\ u'_+ \end{pmatrix} = \mathcal{M} \begin{pmatrix} u_- \\ u'_- \end{pmatrix}; \left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha+m)^2 - \frac{1}{4}}{r^2} \right] u \in L^2(]0,\infty[) \Bigg\} \qquad m \in \mathbb{Z}. \tag{33}$$

Then, let us introduce in $L^2(\mathbb{R}^2)$ the operator

$$H_{\alpha,\mathcal{M}} = \bigoplus_{m=-\infty}^{m=+\infty} U^{-1} h_{\alpha,m,\mathcal{M}_{2\times 2}} U \bigotimes \mathbb{1}. \tag{34}$$

By definition, $H_{\alpha,\mathcal{M}}$ is the rigorous mathematical formulation of the formal expression (1). It provides a slight generalization of H_B.

Finally let us summarize in a theorem some relevant properties generated by the $U(2)$ matrices $\mathcal{M}$ and $\mathcal{N}$.

Theorem 2. *Let*

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{35}$$

and

$$\tau = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{36}$$

Then the matrices $\mathcal{M}$ and $\mathcal{N}$ defined in (21) and (26)–(29) satisfy the following properties:

(i) $\sigma\mathcal{M}\sigma = -\mathcal{N}^t$ *and* $\sigma\mathcal{N}\sigma = -\mathcal{M}^t$*;*
(ii) $\sigma\mathcal{M}\sigma\mathcal{M}^\dagger = (\det\mathcal{M})\sigma\mathcal{N}\sigma\mathcal{N}^t = (\det\mathcal{N})\mathbb{1}$*;*
(iii) $(\sigma\mathcal{M}\sigma\mathcal{M}^\dagger)(\sigma\mathcal{N}\sigma\mathcal{N}^\dagger) = (-\det\mathcal{M})\mathbb{1}\ (-\det\mathcal{N})\mathbb{1} = \mathbb{1}$*;*
(iv) $(\tau\mathcal{M}\tau)(\tau\mathcal{M}^\dagger\tau) = \mathbb{1}$*;*
(v) $\tau\tau = \sigma$.

The statement (iv) shows that the matrix $\mathcal{U} := \tau\mathcal{M}\tau$ is a unitary matrix: $\mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$ with $\mathcal{U}^\dagger = \tau\mathcal{M}^\dagger\tau$.

To conclude this letter, let us note that the complete analysis of spectral and scattering properties of this model requires further technical work and will be thoroughly discussed in a forthcoming paper.

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